I. OPPENHEIMER-VOLKOV SOLUTION IN ISOTROPIC COORDINATES

The Oppenheimer-Volkov solution is usually derived in Schwarzschild coordinates. In these coordinates, the metric can be brought into the form

$$ds^2 = -e^{2\phi}dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{1}$$

The equation of hydrostatic equilibrium is then

$$\frac{dP}{dr} = -(\rho + P)\frac{m + 4\pi r^3 P}{r^2(1 - 2m/r)}, \tag{2}$$

the mass m satisfies

$$m(r) = \int_0^r r^2 \rho dr, \tag{3}$$

and the metric function ϕ

$$\frac{d\phi}{dr} = \frac{m + 4\pi r^3 P}{r^2(1 - 2m/r)}. \tag{4}$$

At the surface of the star, $r = R$, the solution can be matched to the vacuum Schwarzschild solution, which determines a boundary condition for ϕ:

$$\phi(R) = \frac{1}{2}\ln(1 - 2M/R), \tag{5}$$

where $M = m(R)$.

In order to impose initial data in cartesian coordinates, we want to transform this solution to isotropic coordinates, in which the metric takes the form

$$ds^2 = -e^{2\phi}dt^2 + e^{2\mu}(d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{6}$$

Here $\bar{r}$ denotes the isotropic radius. Matching the two metrices, we obviously find

$$r^2 = e^{2\mu}\bar{r}^2 \tag{7}$$

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 = e^{2\mu} d\bar{r}^2 \tag{8}$$

Combining the two equations, we find the differential equation

$$\frac{d\bar{r}}{\bar{r}} = \left(1 - \frac{2m}{r}\right)^{-1/2} \frac{dr}{r} \tag{9}$$

Given a solution $\bar{r}(r)$, the metric potential e^{μ} is then given by

$$e^{2\mu} = \frac{r}{\bar{r}}. \tag{10}$$

We now solve equation (9) separately for the exterior and the interior of the star.

A. Exterior

In the exterior of the star, $r > R$, the mass m is constant, and equation (9) can be solved analytically up to a constant of integration (see Problem 15.13 in the Problem Book, and Problem 3.4 in the Lecture Notes). Fixing this constant such that r and $\bar{r}$ agree at infinity, we find

$$\bar{r} = \frac{1}{2}\left(\sqrt{r^2 - 2Mr} + r - M\right), \tag{11}$$

or, solving for r,

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2. \tag{12}$$

The metric potential e^{μ} as a function of $\bar{r}$ is obviously

$$e^{2\mu} = \left(1 + \frac{M}{2\bar{r}}\right)^2. \tag{13}$$

B. Interior

In the interior, equation (9) can not be integrated analytically, because m, according to (3), is now a function of r. Instead, we have to integrate

$$\int_0^{\bar{r}} \frac{d\bar{r}}{\bar{r}} = \int_0^r \left(1 - \frac{2m}{r}\right)^{-1/2} \frac{dr}{r}. \tag{14}$$

Obviously, the left hand side can be integrated analytically, and has a singular point at $\bar{r} = 0$. The right hand side cannot be integrated analytically, but will also be singular at $r = 0$, which poses a problem when trying to integrate this equation numerically. We therefore rewrite the right hand side by adding and subtracting a term $1/r$, which yields

$$\int_0^r \frac{1}{r(1-2m/r)^{1/2}} dr = \int_0^r \frac{1-(1-2m/r)^{1/2}}{r(1-2m/r)^{1/2}} dr + \int_0^r \frac{dr}{r}. \tag{15}$$

Since $m \sim r^3$ close to the origin, the first term on the right hand side is now regular, and the second one can be integrated analytically. Inserting this into (14), we find

$$\int_0^{\bar{r}} d\ln\bar{r} - \int_0^r d\ln r = \int_0^r \frac{1-(1-2m/r)^{1/2}}{r(1-2m/r)^{1/2}} dr. \tag{16}$$

Replacing the lower limits ($r = \bar{r} = 0$) temporarily with r_0 and $\bar{r}_0$, we can integrate the right hand side and find

$$\ln\bar{r} - \ln\bar{r}_0 - \ln r - \ln r_0 = \ln(\bar{r}/r) - \ln(\bar{r}_0/r_0) = \int_0^r \frac{1-(1-2m/r)^{1/2}}{r(1-2m/r)^{1/2}} dr, \tag{17}$$

or

$$\bar{r} = C r \exp\Big[\int_0^r \frac{1-(1-2m/r)^{1/2}}{r(1-2m/r)^{1/2}} dr\Big]. \tag{18}$$

Here the constant of integration is related to the ratio $\bar{r}_0/r_0$ evaluated at the origin (which is perfectly regular). It can be chosen such that the interior solution matches the exterior solution at the surface of the star. From (11) and (18) we therefore find

$$C = \frac{1}{2R}\Big(\sqrt{R^2 - 2MR} + R - M\Big) \exp\Big[-\int_0^R \frac{1-(1-2m/r)^{1/2}}{r(1-2m/r)^{1/2}} dr\Big]. \tag{19}$$

C. Assigning Initial Data

We now have to assign initial data for the three dimensional evolution. To construct a specific model, we integrate equations (3), (2), (4) and (18) for a given value of the central density ρ_c. Since we are constructing a static solution, the extrinsic curvature is identically zero, and so is the shift. The lapse can be constructed from the metric potential

$$\alpha = e^{\phi}, \tag{20}$$

the conformally related metric is flat,

$$\tilde{\gamma}_{ij} = \eta_{ij} \tag{21}$$

(in cartesian coordinates), and the conformal factor can be constructed from $e^{4\phi} = e^{2\mu}$,

$$\phi = \frac{1}{2} \ln e^{\mu} = \frac{1}{2} \ln(r/\bar{r}). \tag{22}$$

The matter sources can be constructed from the stress-energy tensor

$$T^{\alpha\beta} = (\rho + P) u^{\alpha} u^{\beta} + P g^{\alpha\beta}. \tag{23}$$

Here the four velocity u^{α} is identical to the normal vector n^{α}:

$$u^{\alpha} = n^{\alpha} = (-e^{\phi}, 0, 0, 0). \tag{24}$$

The matter source ρ_N is then

$$\rho_N \equiv n_{\alpha} n_{\beta} T^{\alpha\beta} = \rho. \tag{25}$$

The momentom source S_i vanishes identically

$$S_i \equiv -\gamma_{i\alpha} n_{\beta} T^{\alpha\beta} = g_{i\alpha} n^{\alpha} \rho = 0. \tag{26}$$

The spatial stress tensor S_{ij} is

$$S_{ij} \equiv \gamma_{i\alpha} \gamma_{j\beta} T^{\alpha\beta} = P \gamma_{ij} = P e^{4\phi} \tilde{\gamma}_{ij}, \tag{27}$$

and its trace is obviously

$$S = 3P. \tag{28}$$

The trace free part of S_{ij} therefore vanishes identically.