Version 1.3

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1   Introduction

Gravitational waves (GW) are usually described by a simmetric tensor $h_{ij}(t)$ defined in the transverse-traceless gauge (TT). In the coordinate wave frame, the tensor assumes a simple form characterized by only two components, h_+ and $h_\times$.

The IFO's detector response to GW depends on the orientation of the two arms with respect to the wave incoming direction and it is well described by two quantities: the antenna patterns $(F_+, F_\times)$, defined as the product between the detector tensor D and the wave tensor $(h_+, h_\times)$, as in the following:

$$\begin{bmatrix} F_+ \\ F_\times \end{bmatrix} = \begin{bmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{bmatrix} \begin{bmatrix} h_+ \\ h_\times \end{bmatrix} \tag{1}$$

Using antenna patterns we can define the detector response to a gravitational signal ξ as:

$$\xi = F_+ h_+ + F_\times h_\times \tag{2}$$

To simplify notations, we can define complex waveforms and antenna patterns as:

$$u = h_+ + i h_\times \tag{3}$$

$$A = F_+ + i F_\times \tag{4}$$

where i is the imaginary unity. Using this notation the detector response becomes:

$$\xi = u \cdot \tilde{A} + \tilde{u} \cdot A \tag{5}$$

and a gauge transformation in the wave coordinate frame is described by a rotation of the type:

$$\begin{aligned} A' &= e^{i\psi} A \\ u' &= e^{i\psi} u \end{aligned} \tag{6}$$

hence, a gauge transformation does not change the detector response.

1

2 Likelihood analysis

2.1 A single detector

For our considerations we can use hypothesis test. This suggests to define a *decision rule* to select one of two mutually exclusive hypotheses: the absence (H_0, null) and the presence (H_1, alternative) of signals in the data stream.

Consider $\mathbf{x} = \{x[1], x[2], ..., x[I]\}$ as the detector data, the H_0 and H_1 are described by the two probability densities $p(x|H_0)$ and $p(x|H_1)$. Any decision rule is associated to a threshold applied to these densities, and it is characterized by the table below.

		Signal presence	
		Yes	No
Hypothesis	Yes	True Alarm	False Alarm
	No	False Dismissal	True Dismissal

Table 1: *Right* and *wrong* *possible outcomes from decision rules. Each outcome has an associated probability.*

The Neyman-Pearson criterion says that in the case that the H_1 hypothesis is a simple one, the optimal decision rule has the least false dismissal probability for fixed false alarm probability, i.e. the rule rejects H_0 when the *likelihood ratio* $\Lambda(x)$ is greater than a threshold value fixed by the specified false alarm probability, $\Lambda(x)$ defined as:[1]

$$\Lambda(x) = \frac{p(x|H_1)}{p(x|H_0)} \tag{7}$$

In the GW data analysis, if considering a Gaussian white noise with zero mean, the probabilities densities are:

$$p(x|H_0) = \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{x^2[i]}{2\sigma^2}\right) \tag{8}$$

$$p(x|H_1) = \prod_{i=1}^{I} \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x[i]-\xi[i])^2}{2\sigma^2}\right) \tag{9}$$

where σ is the standard deviation of the noise.

We use the logaritmic value of the likelihood ratio, which we call *likelihood functional* (o simply likelihood):

$$L = ln(\Lambda(x)) = \sum_{i=1}^{I} \frac{1}{\sigma^2}\left(x[i]\xi[i] - \frac{1}{2}\xi^2[i]\right) \tag{10}$$

---

[1]In the case that H_1 is a multiple hypothesis, we cannot use this approach. However, Neymann-Pearson have demonstrated that we can substitute the likelihood ratio with the ratio of maxima among the two hypothesis distributions.

2.2 A network of detectors

The same formalism can be extended to a network of N detectors. So, detector response of the k detector is given by:

$$\xi = u_k \cdot \tilde{A}_k + \tilde{u}_k \cdot A_k \tag{11}$$

and the total likelihood becomes:

$$L = \sum_{k=1}^{N} \sum_{i=1}^{I} \frac{1}{\sigma_k^2} \left(x_k[i]\xi_k[i] - \frac{1}{2}\xi_k^2[i] \right) \tag{12}$$

To simplify notations, it is useful to use a N-dimensional space, where each dimension represents a particular detector. In this space, each vector has N components. So:

$$
\begin{aligned}
F_+ &= \{F_{1+}, F_{2+}, ..., F_{N+}\} \\
F_\times &= \{F_{1\times}, F_{2\times}, ..., F_{N\times}\} \\
\sigma &= \{\sigma_1, \sigma_2, .../\sigma_N\} \\
u &= \{u_1, u_2, ..., u_N\} \\
A &= \{A_1, A_2, ..., A_N\}
\end{aligned} \tag{13}
$$

We can also define the network antenna patterns:

$$g_r = \sum_{k=1}^{N} \frac{A_k \cdot \tilde{A}_k}{\sigma_k^2} \tag{14}$$

$$g_c = \sum_{k=1}^{N} \frac{A_k^2}{\sigma_k^2} \tag{15}$$

We focus on the transformation which allows that the imaginary part of g_c is null:s i.e. if $g_c = |g_c|e^{2i\gamma}$ the transformation is: $A_k' = A_k e^{-i\gamma}$.

$$
\begin{aligned}
F_{k+}' &= F_{k+}Cos(\gamma) + F_{k\times}Sin(\gamma) \\
F_{k\times}' &= -F_{k+}Sin(\gamma) + F_{k\times}Cos(\gamma)
\end{aligned} \tag{16}
$$

We can see that the imaginary part of g_c' equal to zero is equivalent to the orthogonality of the two vectors: $f_+' = \{\frac{F_{1+}'}{\sigma_1}, ..., \frac{F_{N+}'}{\sigma_N}\}$ and $f_\times' = \{\frac{F_{1\times}'}{\sigma_1}, ..., \frac{F_{N\times}'}{\sigma_N}\}$, infact if we define the vector $A_s = \{\frac{A_1}{\sigma_1}, ..., \frac{A_N}{\sigma_N}\}$:

$$
\begin{aligned}
A_s'^2 = A_s' \cdot A_s' &= \\
&= (f_+' + i \cdot f_\times') \cdot (f_+' + i \cdot f_\times') = \\
&= f_+'^2 - f_\times'^2 + 2i(f_+' \cdot f_\times')
\end{aligned} \tag{17}
$$

We call this frame **Dominant Polarization Frame (DPF)**.

From the previous equation we can define the transformation from a general frame $(f_+, f_\times)$ to the DPF frame $(f'_+, f'_\times)$, i.e. to define the angle γ as a function of $(f_+, f_\times)$.

$$f_+^2 - f_\times^2 + 2i(f_+ \cdot f_\times) = A_s^2 = |A_s|e^{i\gamma}|A_s|e^{i\gamma} = |A_s|^2 e^{2i\gamma} = |A_s|^2[Cos(2\gamma) + iSin(2\gamma)] \quad (18)$$

From this we have:
$$\begin{aligned} Cos(2\gamma)|A_s|^2 &= (f_+^2 - f_\times^2) \\ Sin(2\gamma)|A_s|^2 &= 2(f_+ \cdot f_\times) \end{aligned} \quad (19)$$

These are the transformation from the frame $(f_+, f_\times)$ to the DPF frame. We should calculate from these equalities the expression of $Cos(\gamma)$ and $Sin(\gamma)$, but we are not interested in it for the following considerations.

We can easily see that these transformations assure the orthogonality of f_+ and $f_\times$ vectors. Infact:

$$\begin{aligned} f'_+ \cdot f'_\times &= (f_+ Cos(\gamma) + f_\times Sin(\gamma)) \cdot (f_+ Sin(\gamma) + f_\times Cos(\gamma)) = \\ &= -(f_+^2 - f_\times^2)Cos(\gamma)Sin(\gamma) + (f_+ \cdot f_\times)(Cos^2(\gamma) - Sin^2(\gamma)) = \\ &= -(f_+^2 - f_\times^2)Sin(2\gamma)/2 + (f_+ \cdot f_\times)Cos(2\gamma) = 0 \end{aligned} \quad (20)$$

We can calculate see that the DPF antenna patterns are related to the network antenna patterns (Eq: 14 and 15).

$$\begin{aligned} |f'_+|^2 &= |f_+ Cos(\gamma) + f_\times Sin(\gamma)|^2 = \\ &= f_+^2 Cos^2(\gamma) + f_\times^2 Sin^2(\gamma) + (f_+ \cdot f_\times)Cos(\gamma)Sin(\gamma) = \\ &= f_+ \frac{1 + Cos(2\gamma)}{2} + f_\times \frac{1 - Cos(2\gamma)}{2} + (f_+ \cdot f_\times)\frac{Sin(2\gamma)}{2} = \\ &= \frac{1}{2}((f_+^2 + f_\times^2) + |A_s^2|) = \frac{1}{2}(|A| + |A_s^2|) \end{aligned} \quad (21)$$

E analogamente per $|f'_\times|^2$

$$|f'_\times|^2 = \frac{1}{2}(|A| - |A_s^2|) \quad (22)$$

Into the DPF, the likelihood assumes a simple form.[2]
If we define the normalized vector: $X = \{\frac{x_1}{\sigma_1}, ..., \frac{x_n}{\sigma_N}\}$ and $\Xi = \{\frac{\xi_1}{\sigma_1}, ..., \frac{\xi_n}{\sigma_N}\}$, the likelihood becomes:

$$\begin{aligned} L &= \left(X \cdot \Xi - \frac{1}{2}\Xi \cdot \Xi \right) = \\ &\left[X \cdot (f'_+ h_+ + f'_\times h_\times) - \frac{1}{2}(f'_+ h_+ + f'_\times h_\times) \cdot (f'_+ h_+ + f'_\times h_\times) \right] = \\ &\left[X \cdot f'_+ h_+ + X \cdot f'_\times h_\times - \frac{1}{2}(f'^2_+ h_+^2 + f'^2_\times h_\times^2) \right] \end{aligned} \quad (23)$$

---

[2]In the following equations we do not consider the $\sum_i$, but it should be still present.

where the last equality holds only in the DPF. The maximum likelihood is calculated from $\frac{\delta L}{\delta h_+} = 0$ and $\frac{\delta L}{\delta h_\times} = 0$:

$$\begin{aligned} \frac{\delta L}{\delta h_+} &= X \cdot f'_+ - |f'_+|^2 h_+ = 0 \\ \frac{\delta L}{\delta h_\times} &= X \cdot f'_\times - |f'_\times|^2 h_\times = 0 \end{aligned} \tag{24}$$

which gives the solutions:

$$\begin{aligned} h_+ &= \frac{X \cdot f'_+}{|f'_+|^2} \\ h_\times &= \frac{X \cdot f'_\times}{|f'_\times|^2} \end{aligned} \tag{25}$$

The maximum likelihood is given by these solutions to the equation:

$$\begin{aligned} L &= \left[X \cdot f'_+ h_+ + X \cdot f'_\times h_\times - \frac{1}{2}(f'^2_+ h^2_+ + f'^2_\times h^2_\times) \right] = \\ &\left[X \cdot f'_+ \frac{X \cdot f'_+}{|f_+|^2} + X \cdot f'_\times \frac{X \cdot f'_\times}{|f'_\times|^2} - \frac{1}{2}\left(f'^2_+ \frac{(X \cdot f'_+)^2}{|f_+|^4} + f'^2_\times \frac{(X \cdot f'_\times)^2}{|f'_\times|^4} \right) \right] = \\ &\left(\frac{(X \cdot f'_+)^2}{|f'_+|^2} + \frac{(X \cdot f'_\times)^2}{|f'_\times|^2} \right) \end{aligned} \tag{26}$$

This is equivalent to project the **x** vector on the plane defined by f'_+ and $f'_\times$ (see Fig. 1). We define as u the projection of X in the f'_+ and $f'_\times$ plane.

$$u = f'_+ \frac{X \cdot f'_+}{|f'_+|} + f'_\times \frac{X \cdot f'_\times}{|f'_\times|} \tag{27}$$

However, it is not necessary to use te DPF to project **x** in the f'_+ - $f'_\times$ plane. Infact from Eq. 16 we see that the planes defined from f'_+ - $f'_\times$ and f_+ - $f_\times$ are the same, because the vectors that generate them are related by a linear composition. So, the **u** vector is the same in the two frames. Unfortunately, in the starting frame f_+ and $f_\times$ are not orthogonal, so we cannot use the quantity $X \cdot f_+$ and $X \cdot f_\times$ to construct the vector **u**. However, we can solve the problems using the curl.

In Fig. 2 we introduce the vector $f_+ \times f_\times$, which describes the sub-space orthogonal to the plane f_+ - $f_\times$. Then we define the vector **v**, $v = X \times (f_+ \times f_\times)$. This vector is on the plane f_+ - $f_\times$ and it is orthogonal to **X**. So, if we define the vector $u' = -v \times (f_+ \times f_\times)$, **u** is on the plane f_+ - $f_\times$ (because it is orthogonal to $f_+ \times f_\times$) and (apart from the module) it is the projection of **X** (because it is othogonal to **v**, which is orthogonal to **X**, see Appendix A for more details). The vector **u'** differ from the likelihood only for the module, so to obtain the likelihood we should project **X** on **u'**, so:

$$L = \frac{(x \cdot u')^2}{|u'|^2} \tag{28}$$

From now we will not distinguish between u and u', because they difference only for the module, and in the likelihood calculation we normalize for the module of the vector u used.
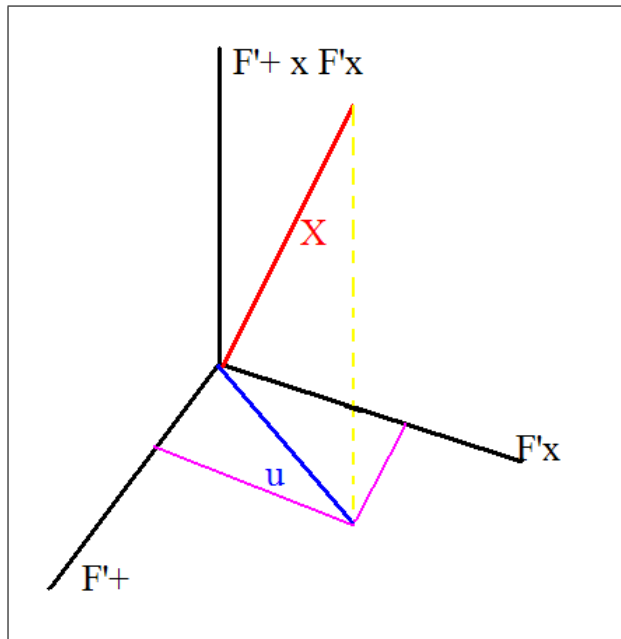
5

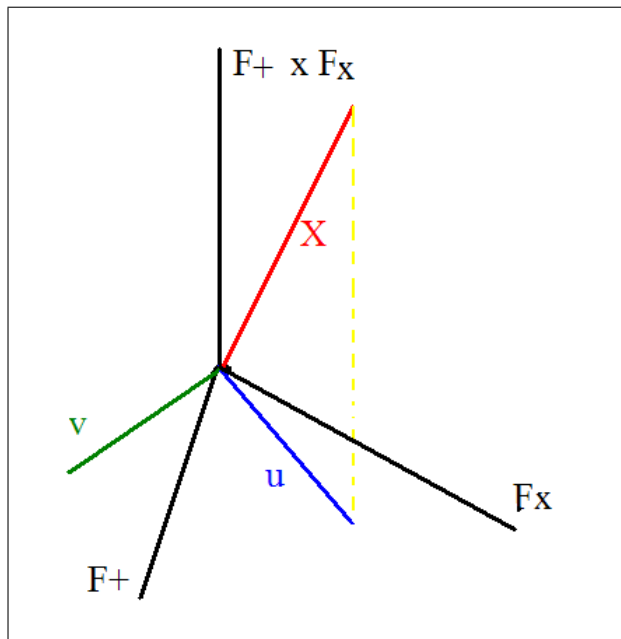Figure 1: *Dominant Polarization Frame*

Figure 2: *General Frame, F_+ and $F_\times$ are not orthogonal*

3 Application

We have many degrees of freedom when analysing data. Because of this, it is possible to obtain unphysical solutions. Now we focus on how to avoid this. We remind the definition on the likelihood:

$$L = \sum_k \left(\frac{x_k^2}{\sigma_k^2} - \frac{(x_k - \xi_k)^2}{\sigma_k^2} \right) \tag{29}$$

We remember the following definitions:

$$f_{k+} = \frac{F_{k+}}{\sigma_k} \qquad f_{k\times} = \frac{F_{k\times}}{\sigma_k} \qquad X = \frac{x_k}{\sigma_k} \tag{30}$$

With this notations the detector response can be written: $\xi = f_+ h_+ + f_\times h_\times$

From previous consideration, we consider the value of $\mathbf{L}$ obtained projecting $\mathbf{X}$ on the F_+ - $F_\times$ plane. We can define two quantities: L_+ e $L_\times$ such as $L = L_+ + L_\times$. In the Dominant Polarization Frame

$$L_+ \sim \frac{(Xf_+)^2}{|f_+|^2} \qquad L_\times \sim \frac{(Xf_\times)^2}{|f_\times|^2} \tag{31}$$

The difference between the two vectors ξ e $\mathbf{X}$ is the detectors noise ($X = \xi + n$). Referring to the Fig. 2, we define the vector h as the projection of $\mathbf{X}$ on $\mathbf{u}$ ($(u \cdot X)/|u|^2 = h$), so for each detector k: $\xi_k = u_k h$.

The following equation is valid:

$$\sum_k <X_k \xi_k> = \sum_k <\xi_k^2> \tag{32}$$

where the second part is the total reconstructed energy. We have also:

$$\sum_k | <X_k \xi_k> - <\xi_k^2> | \neq 0 \tag{33}$$

These informations are useful to avoid unphysical solutions, if we consider the *null stream*:

$$n_k^2 = (X_k - \xi_k)^2 = X_k^2 - \xi_k^2 - 2(X_k \xi_k - \xi_k^2) \tag{34}$$

we see that this quantity is positive defined, but it is not always ensured by the second part of the equation. To fix positivity, we can consider the situation $X_k^2 > \xi_k^2$, but it is better to find the coordinate frame in which the quantity $X_k \xi_k - \xi_k^2$ is minimal. We define the quantity *energy disbalance*:[3]

$$q_k = \frac{X_k \xi_k - \xi_k^2}{h} = X_k u_k - u_k^2 h \tag{35}$$

---

[3]We decide to divide the quantity $X_k \xi_k - \xi_k^2$ to h because in this way we do not consider the solution: $h = 0$

which depends on the **u** vector. We can consider the angle ψ defined by the two vectors ξ and f_+, so $u = u(\psi)$:

$$u = u_0 Cos(\psi) + v_0 Sin(\psi)$$
$$v = -u_0 Sin(\psi) + v_0 Cos(\psi)$$

(36)

where $v = \frac{\partial u}{\partial \psi}$ is orthogonal to **u** if $|u_0| = |v_0| = 1$ (Fig. 2).

We find the minimum of the expression: $\frac{\partial}{\partial \psi}(q \cdot q) = 0$. We define the vector **p** such as $\frac{\partial}{\partial \psi}(q \cdot q) = -2q \cdot p$, i.e.:

$$q_k = X_k u_k - u_k^2 h$$
$$p_k = (2u_k h - X_k)v_k + u_k^2 (X \cdot v)/|u|^2$$

(37)

Infact

$$
\begin{aligned}
p_k &= -\frac{\partial q}{\partial \psi} = -\frac{\partial}{\partial \psi}(X_k u_k - u_k^2 h) \\
&= -[X_k \frac{\partial u_k}{\partial \psi} - 2u_k h \frac{\partial u_k}{\partial \psi} - u_k^2 (X \cdot \frac{\partial u_k}{\partial \psi})] \\
&= -[X_k v_k - 2u_k h v_k - u_k^2 (x \cdot v)] = (2u_k h - X_k)v_k + u_k^2 (X \cdot v)
\end{aligned}
$$

(38)

We normalize the disbalance for the quantity $\frac{X_k}{X_k^2 + \delta}$. In this way the detectors in the network are comparable, otherwise there could be a detector with a big disbalance and a small SNR that could modify consistently the minimum.

If we consider the cases when $\psi \sim 0$, the previous equation can be written (see Appendix B):

$$q \sim q_0 Cos(\psi) - p_0 Sin(\psi)$$
$$p \sim q_0 Sin(\psi) + p_0 Cos(\psi)$$

(39)

so the $q \cdot p = 0$ is satisfied by:

$$Cos(2\psi) = p_0^2 - q_0^2$$
$$Sin(2\psi) = 2(q_0 \cdot p_0)$$

(40)

So we can calculate the maximum likelihood, defined as the projection of **X** on **u** ($L = (X \cdot u)^2/|u|^2$).

4 Regulators

We have defined the maximum likelihood starting from the definition of useful quantities: u, $h = (X \cdot u)/|u|^2$, $\xi_k = h_k$. Starting from these vectors, we find the minimum of energy disbalance to find the correspondence likelihood. The problem of minimum is not easily solvable, because it is possible to have more local minima. We should adopt a criterium to choose the right local minimum. One solution is to use a regulator to define the best guess of u and than find the estimator for the closest minimum.

The use of regulators is defined by the introduction of the two variable quantities $(\delta_+, \delta_\times)$ at the definition of u in the DPF:

$$u = f_+ \frac{X \cdot f_+}{|f_+|^2 + \delta_+} + f_\times \frac{X \cdot f_\times}{|f_\times|^2 + \delta_\times} \propto$$
$$\propto f_+(X \cdot f_+)(|f_\times|^2 + \delta_\times) + f_\times(X \cdot f_\times)(|f_+|^2 + \delta_+) \tag{41}$$

We can use for instance the following regulators:

- **weak**: u is aligned with $\xi \to \delta_+ = 0$, $\delta_\times = 0$

- **soft**: u is between f_+ and $\xi \to \delta_+ = \infty$, $\delta_\times = \infty$

- **hard**: u is aligned with $f_+ \to \delta_+ = 0$, $\delta_\times = \infty$

- **mild**: $u = f_+ \pm f_\times \to \delta_+ \neq 0$, $\delta_\times \neq 0$

5 Appendix A

We explain how to calculate the vector $\mathbf{u}$ in the Fig. 2 in the reference frame $(f_+, f_\times)$ without using the DPF.
Remembering that $u = -v \times (f_+ \times f_\times)$ and the equality:

$$a \times (b \times c) = b(a \cdot c) - c(a \cdot b) \tag{42}$$

we have:

$$u = -v \times (f_+ \times f_\times) = -f_+(v \cdot f_\times) + f_\times(v \cdot f_+) \tag{43}$$

The vector $\mathbf{u}$ is on the plane f_+ - $f_\times$, so we can define it as a linear combination of the two vectors: $u = f_+ u_c + f_\times u_s$. Comparing this equality with the previous equation we have: $u_c = -v \cdot f_\times$ and $u_s = v \cdot f_+$.
Remebering the definition of $\mathbf{v}$:

$$v = X \times (f_+ \times f_\times) = f_+(X \cdot f_\times) - f_\times(X \cdot f_+) = f_+ X_\times - f_\times X_+ \tag{44}$$

where we have defined the quantities: $X_+ = X \cdot f_+$ and $X_\times = X \cdot f_\times$. Then:

$$u_c = -v \cdot f_\times = -(f_+ X_\times - f_\times X_+) \cdot f_\times = X_+(f_\times \cdot f_\times) - X_\times(f_+ \cdot f_\times)$$
$$u_s = v \cdot f_+ = (f_+ X_\times - f_\times X_+) \cdot f_+ = X_\times(f_+ \cdot f_+) - X_+(f_\times \cdot f_+) \tag{45}$$

6 Appendix B

The equation 39 is an approximation for $\psi \sim 0$. We have:

$$u = u_0 Cos(\psi) + v_0 Sin(\psi)$$
$$v = -u_0 Sin(\psi) + v_0 Cos(\psi) = \frac{\partial u}{\partial \psi}$$
$$q_k = X_k u_k - u_k^2 h \tag{46}$$
$$p_k = -\frac{\partial q}{\partial \psi} = (2u_k h - X_k)v_k + u_k^2(X \cdot v)$$

9

To obtain this approximation, we express q as a function of u_0, v_0, $Cos(\psi)$ and $Sin(\psi)$. The k index refer to the detectors involved. We use the conventions previously introduced: quantities with k index are vector components. So the scalar product ($\cdot$) refers to the sum between indexes k.

$$
\begin{aligned}
q =& Xu - u^2 h = \\
& X(u_0 Cos(\psi) + v_0 Sin(\psi)) + \\
& - (u_0 Cos(\psi) + v_0 Sin(\psi))^2 [X \cdot (u_0 Cos(\psi) + v_0 Sin(\psi))]
\end{aligned}
\tag{47}
$$

In the limit $\psi \to 0$, then $Sin^n(\psi) \sim 0$ for $n \geq 2$:

$$
\begin{aligned}
Sin^2(\psi) &\sim 0 \\
Cos^2(\psi) = 1 - Sin^2(\psi) &\sim 1
\end{aligned}
\tag{48}
$$

So, the previous equation is simplified as:

$$
\begin{aligned}
q =& Xu - u^2 h \sim \\
\\
& Xu_0 Cos(\psi) + Xv_0 Sin(\psi) + \\
& - (u_0^2 + 2u_0 v_0 Cos(\psi)Sin(\psi))[(X \cdot u_0)Cos(\psi) + (X \cdot v_0)Sin(\psi)] = \\
\\
& Cos(\psi)[Xu_0 - u_0^2(X \cdot u_0)] + \\
& Sin(\psi)[Xv_0 - u_0^2(X \cdot v_0) - 2u_0 v_0(X \cdot u_0)] = \\
\\
& Cos(\psi)[Xu_0 - u_0^2(X \cdot u_0)] + \\
& - Sin(\psi)[(2u_0 h_0 - X)v_0 + u_0^2(X \cdot v_0)] = \\
\\
& Cos(\psi)q_0 - Sin(\psi)p_0
\end{aligned}
\tag{49}
$$

For p vector we repeat the same procedure.
$p = (2uh - X)v + u^2(X \cdot v) = 2uhv - Xv + u^2(X \cdot v)$, we consider term by term.

$$
\begin{aligned}
uhv =& [u_0 Cos(\psi) + v_0 Sin(\psi)][X \cdot (u_0 Cos(\psi) + v_0 Sin(\psi))][-u_0 Sin(\psi) + v_0 Cos(\psi)] \sim \\
& [(X \cdot u_0)Cos(\psi) + (X \cdot v_0)Sin(\psi)][(-u_0^2 + v_0^2)Cos(\psi)Sin(\psi) + u_0 v_0] = \\
& (X \cdot u_0)(-u_0^2 + v_0^2)Sin(\psi) + (X \cdot u_0)u_0 v_0 Cos(\psi) + (X \cdot v_0)u_0 v_0 Sin(\psi)
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
u^2(X \cdot v) =& (u_0 Cos(\psi) + v_0 Sin(\psi))^2 [X \cdot (-u_0 Sin(\psi) + v_0 Cos(\psi))] \sim \\
& (u_0^2 + 2u_0 v_0 Cos(\psi)Sin(\psi))(-(X \cdot u_0)Sin(\psi) + (X \cdot v_0)Cos(\psi)) = \\
& - u_0^2(X \cdot u_0)Sin(\psi) + u_0^2(X \cdot v_0)Cos(\psi) + 2u_0 v_0(X \cdot v_0)Sin(\psi)
\end{aligned}
\tag{51}
$$

and adding the terms:

$$p = 2uhv - Xv + u^2(X \cdot v) \sim$$

$$2[(X \cdot u_0)(-u_0^2 + v_0^2)Sin(\psi) + (X \cdot u_0)u_0v_0Cos(\psi) + (X \cdot v_0)u_0v_0Sin(\psi)]+$$
$$- [-Xu_0Sin(\psi) + Xv_0Cos(\psi)]+$$
$$- u_0^2(X \cdot u_0)Sin(\psi) + u_0^2(X \cdot v_0)Cos(\psi) + 2u_0v_0(X \cdot v_0)Sin(\psi) =$$

$$= Sin(\psi)[2(X \cdot u_0)(-u_0^2 + v_0^2) + 2(X \cdot v_0)u_0v_0 + Xu_0 - u_0^2(X \cdot u_0) + 2u_0v_0(X \cdot v_0)]$$
$$Cos(\psi)[2(X \cdot u_0)u_0v_0 - Xv_0 + u_0^2(X \cdot v_0)] =$$

$$= Sin(\psi)q_0 + Cos(\psi)p_0 + \text{small terms...}$$

(52)

Starting from the equations:

$$q = q_0Cos(\psi) - p_0Sin(\psi)$$
$$p = q_0Sin(\psi) + p_0Cos(\psi)$$

(53)

We can find the minimum:

$$q \cdot p = (q_0Cos(\psi) - p_0Sin(\psi))(q_0Sin(\psi) + q_0Cos(\psi)) =$$
$$(q_0^2 - p_0^2)Sin(\psi)Cos(\psi) + (q_0 \cdot p_0)[Cos^2(\psi) - Sin^2(psi)] =$$
$$(q_0^2 - p_0^2)Sin(2\psi)/2 + (q_0 \cdot p_0)Cos(2\psi)$$

(54)

so $q \cdot p = 0$ is satisfied by:

$$Cos(2\psi) = p_0^2 - q_0^2$$
$$Sin(2\psi) = 2(q_0 \cdot p_0)$$

(55)

7   Appendix C

The role of regulators is to decide a position in the $(F'_+, F'_\times)$ DPF plane to use as a starting point to calculate the likelihood value. From this starting point, the algorithm calculate the position in the plane which has the minimum value of energy disbalance. We have just defined the regulators in the DPF, moreover we can find the expression of u for each regulator also in the general frame $(F_+, F_\times)$. In general we can express u in the following form:

$$u = f_+u_c + f_\times u_s$$

(56)

In the Appendix A we have calculated the u vector in the **weak** definition, which assumes that the u vector is aligned with ξ, and consequently with X.

$$u_c(weak) = -v \cdot f_\times = -(f_+X_\times - f_\times X_+) \cdot f_\times = X_+(f_\times \cdot f_\times) - X_\times(f_+ \cdot f_\times)$$
$$u_s(weak) = v \cdot f_+ = (f_+X_\times - f_\times X_+) \cdot f_+ = X_\times(f_+ \cdot f_+) - X_+(f_\times \cdot f_+)$$

(57)

We can calculate the **soft** definition of u from this equation. Infact the soft can be obtained assuming that $\delta_+, \delta_\times \to \infty$. This means that in the previous definition only the first term is important.

$$u_c(soft) = \lim_{\delta_+, \delta_\times \to \infty} u_c(weak) = X_+$$
$$u_s(soft) = \lim_{\delta_+, \delta_\times \to \infty} u_s(weak) = X_\times \tag{58}$$

The **hard** definition of u can be found taking in account that we are interested only in the u direction and not in the module. So for this regulator $u = f'_+$, but from Eq. 16 we have that $f'_+ = f_+ Cos(\gamma) + f_\times Sin(\gamma)$. Because we are interested only in direction, we calculate the quantity $Cos(\gamma)u$ instead of u. So ,we have:

$$Cos(\gamma)u = f_+ Cos^2(\gamma) + f_\times Sin(\gamma)Cos(\gamma) = f_+ \frac{1}{2}[1 + Cos(2\gamma)] + f_\times \frac{1}{2}Sin(2\gamma) \tag{59}$$

and from Eq. 19 we have:

$$Cos(\gamma)u = f_+ \frac{1}{2}\left[1 + \frac{f_+^2 - f_\times^2}{|A|^2}\right] + f_\times \frac{f_+ \cdot f_\times}{|A|^2} \tag{60}$$

So, multiplying by $|A|^2 = |A^2|$ and discarding about the factors on the left part we have the expression for u_c and u_s:

$$u_c(hard) = \frac{1}{2}|A|^2 + (f_+^2 - f_\times^2)$$
$$u_s(hard) = f_+ \cdot f_\times \tag{61}$$

Finally the mild regulator defines $u = f'_+ \pm f'\times$. We can repeat the same procedure for the hard regulator.

$$Cos(\gamma)u = Cos(\gamma)(f'_+ \pm f'_\times) =$$
$$= (f_+ Cos^2(\gamma) + f_\times Cos(\gamma)Sin(\gamma)) \pm (-f_+ Cos(\gamma)Sin(\gamma) + f_\times Cos^2(\gamma)) =$$
$$= f_+(Cos^2(\gamma) \mp Cos(\gamma)Sin(\gamma)) + 1/2 f_\times (Sin(\gamma)Cos(\gamma) \pm 2Cos^2(\gamma)) =$$
$$= f_+ \left[\frac{1}{2}\left(1 + \frac{f_+^2 - F_\times^2}{|A|^2}\right) \mp \frac{f_+ \cdot f_\times}{|A|^2}\right] + f_\times \left[\frac{f_+ \cdot f_\times}{|A|^2} \pm \frac{1}{2}\left(1 + \frac{f_+^2 - f_\times^2}{|A|^2}\right)\right] \tag{62}$$

Repeating the procedure of multipling for $|A|^2$ and discarding for the common factor, we obtain the final form:

$$u_c(mild) = 1/2|A|^2 + 1/2(f_+^2 - f_\times^2) \mp (f_+ \cdot f_\times)$$
$$u_s(mild) = (f_+ \cdot f_\times) \pm 1/2(|A|^2 - (f_+^2 - f_\times^2)) \tag{63}$$

The $\pm$ sign depends on what quadrants of the $f_+, f_\times$ plane the X vector is projected. So, if the porjection of X is on the first or third quadrant the sign is $+$, otherwise ths sign is $-$. This conditions can be expressed using the disuguagliances:

$$\begin{array}{cc} \text{Condition} & \text{Sign} \\ (X \cdot f'_+)(X \cdot f'_\times) > 0 & + \\ (X \cdot f'_+)(X \cdot f'_\times) < 0 & - \end{array} \tag{64}$$

It is possible to use choose intermediate situations between regulators using variable parameters. For instance in the definition of mild regulators: $u_c(mild) = 1/2|A|^2 + 1/2(f_+^2 - f_\times^2) \mp (f_+ \cdot f_\times)$ we can define a parameter called *mild* in the same way: $u_c(mild) = 1/2|A|^2 + 1/2(f_+^2 - f_\times^2) \mp mild(f_+ \cdot f_\times)$. Changing the value of the parameter *mild* from 0 to 1 we can use an intermediate regulator from mild (mild=1) to hard (mild=0) definitions. We can use one definition of u_c and u_s for all the regulators in the following way:

$$
\begin{aligned}
u_c(all) = {}& \text{soft} X_+ + \\
& + \text{mild}(f_+ \cdot f_\times)\text{gg} + \\
& + \frac{\text{hard}}{(f_+^2 + f_\times^2)/2 + |A|^2} \left[X_+ \left(|A^2| + \frac{f_+^2 - f_\times^2}{2} \right) + X_\times(f_+ \cdot f_\times) \right] \\
u_s(all) = {}& \text{soft} X_\times + \\
& + \text{mild}(|A^2| + (f_+^2 - f_\times^2)/2)\text{gg} + \\
& + \frac{\text{hard}}{(f_+^2 + f_\times^2)/2 + |A|^2} \left[X_\times \left(|A|^2 - \frac{f_+^2 - f_\times^2}{2} \right) + X_+(f_+ \cdot f_\times) \right]
\end{aligned}
\tag{65}
$$

where gg is defined as:

$$
gg = \frac{1}{\left(|A|^2 + \frac{f_+^2 - f_\times^2}{2} \right)^2 + (f_+ \cdot f_\times)^2}
$$
$$
\left\{ \left[X_\times \left(|A^2| + \frac{f_+^2 - f_\times^2}{2} \right) - X_+(f_+ \cdot f_\times) \right] \pm \left| X_+ \left(|A^2| + \frac{f_+^2 - f_\times^2}{2} \right) + X_\times(f_+ \cdot f_\times) \right| \right\}
\tag{66}
$$

We can obtain the previous definitions, apart from common factors (but we know that we are interested in direction and not in the module of u) using the following parameters values:

- **soft**: soft=1, mild=0, hard=0;

- **weak**: soft=1, mild=0, hard=-1;

- **mild**: soft=1, mild=1, hard=0;

- **hard**: soft=0, mild=0, hard=1;

8 Appendix D

In this section we describe the corrispondence between the formula previous used and the code, in particular the *likelihoodB* function in the *network* class. In the code the normalized antenna patterns are indicated with Fp and Fx, where as the normalized

data vector X is called am.

$$f_+ = \{\frac{F'_{1+}}{\sigma_1}, ..., \frac{F'_{N+}}{\sigma_N}\} = \text{Fp}$$
$$f_\times = \{\frac{F'_{1\times}}{\sigma_1}, ..., \frac{F'_{N\times}}{\sigma_N}\} = \text{Fx} \qquad (67)$$
$$X = \{\frac{x}{\sigma_1}, ..., \frac{x}{\sigma_N}\} = \text{am}$$

Scalar products between these vectors are called with the following labels:

$$\text{gp} = f_+ \cdot f_+ = f_+^2$$
$$\text{gx} = f_\times \cdot f_\times = f_\times^2$$
$$\text{gI} = f_+ \cdot f_\times \qquad (68)$$
$$\text{Xp} = X \cdot f_+ = X_+$$
$$\text{Xx} = X \cdot f_\times = X_\times$$

The code introduces the following values:

- $|A_s|/2$: gr = (gp+gx)/2

- real part of $A_s/2$: gR = (gp-gx)/2

- $|(A_s/2)^2| = |A_s/2|^2$: gc = $\sqrt{\text{gR}^2 + \text{gI}^2}$

The code defines also the transformation angles to DPF frame:

$$\text{co} = Cos^2(\gamma)/2 = Cos(\gamma)Cos(\gamma)/2 = \text{gc+gR}$$
$$\text{si} = Sin(2\gamma) = Sin(\gamma)Cos(\gamma)/2 = \text{gI} \qquad (69)$$

where we have multiplied the angles for the common factor $Cos(\gamma)/2$ because we are interested in directions of vectors, and not in the module, s in the previous paragraph.

Now we can write the definitions of u_s and u_s as in the code:

$$u_c = \text{soft*Xp} + \text{mild*si*gg} + \frac{\text{hard}}{\text{gr+gc}} * (\text{Xp(gc+gR)+Xx*gI})$$
$$u_s = \text{soft*Xx - mild*co*gg} + \frac{\text{hard}}{\text{gr+gc}} * (\text{Xx*(gc-gR)+Xp*gI}) \qquad (70)$$

where gg is defined as:

$$\text{gg} = \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xx*co-Xp*si} \pm |\text{Xp*co+Xx*si}|) \qquad (71)$$

Now we show how to obtain the different regulators from these equations.

Soft regulator
Reminding the definition of soft regulator, and writing it in the code format:

$$u_c(soft) = X_+ = \text{Xp}$$
$$u_s(soft) = X_\times = \text{Xx} \qquad (72)$$

that can be obtained from Eq. 70 assuming soft=1, mild=0, hard=0.

Weak regulator

Reminding the definition of weak regulator, and writing it in the code format:

$$u_c(weak) = X_+(f_\times \cdot f_\times) - X_\times(f_+ \cdot f_\times) = \text{Xp*gx - Xx*gI}$$
$$u_s(weak) = X_\times(f_+ \cdot f_+) - X_+(f_\times \cdot f_+) = \text{Xx*gp-Xp*gI}$$

(73)

that can be obtained from Eq. 70 assuming soft=1, mild=0, hard=-1

$$
\begin{aligned}
u_c &= \text{Xp} - \frac{1}{\text{gr+gc}} * (\text{Xp(gc+gR)+Xx*gI}) = \\
&= \frac{1}{\text{gr+gc}} * (\text{Xp*(gc+gr-gc-gR)-Xx*gI}) = \\
&= \frac{1}{\text{gr+gc}} * (\text{Xp*gx-Xx*gI}) \\
u_s &= \text{Xx} - \frac{1}{\text{gr+gc}} * (\text{Xx*(gc-gR)+Xp*gI}) = \\
&= \frac{1}{\text{gr+gc}} * (\text{Xx*(gr+gc-gc+gR)-Xp*gI}) = \\
&= \frac{1}{\text{gr+gc}} * (\text{Xx*gp-Xp*gI})
\end{aligned}
$$

(74)

that is the soft definition apart from a common factor: $\frac{1}{\text{gr+gc}}$.

Mild regulator

Reminding the definition of mild regulator, and writing it in the code format:

$$u_c(mild) = 1/2|A|^2 + 1/2(f_+^2 - f_\times^2) \mp (f_+ \cdot f_\times) = \text{(gc+gR)} \pm \text{gI}$$
$$u_s(mild) = (f_+ \cdot f_\times) \pm 1/2(|A|^2 - (f_+^2 - f_\times^2)) = \text{gI} \pm \text{(gc+gR)}$$

(75)

that can be obtained from Eq. 70 assuming soft=1, mild=1, hard=0

$$u_c = \text{Xp} + \text{si*gg} =$$

$$= \text{Xp+si} * \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xx*co-Xp*si} \pm |\text{Xp*co+Xx*si}|) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xp*(co*co+si*si)+Xx*co*si-Xp*si*si} \pm \text{si} * |\text{Xp*co+Xx*si}|) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xp*co+Xx*si}) * (\text{co} \pm \text{si}) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xp*co+Xx*si}) * (\text{gc+gR} \pm \text{gI})$$

$$u_s = \text{Xx - co*gg} = \tag{76}$$

$$= \text{Xx - co} * \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xx*co-Xp*si} \pm |\text{Xp*co+Xx*si}|) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xx*(co*co+si*si)-Xx*co*co+Xp*si*co} \pm \text{co} * |\text{Xp*co+Xx*si}|) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xp*co+Xx*si}) * (\text{si} \pm \text{co}) =$$

$$= \frac{1}{\text{co}^2 + \text{si}^2}(\text{Xp*co+Xx*si}) * (\text{gI} \pm \text{gc+gR})$$

that is the mild definition apart from a common factor: $\frac{1}{\text{co}^2+\text{si}^2}(\text{Xp*co+Xx*si})$.

Hard regulator

Reminding the definition of hard regulator, and writing it in the code format:

$$u_c(hard) = 1/2|A|^2 + 1/2(f_+^2 - f_\times^2) = \text{gc+gR}$$
$$u_s(hard) = f_+ \cdot f_\times = \text{gI} \tag{77}$$

that can be obtained from Eq. 70 assuming soft=0, mild=0, hard=1.
Disregarding the common factor $\frac{1}{\text{gr+gc}}$, it is convenient to multiply the two quantities for (gc+gR)

$$(\text{gc+gR}) * u_c = (\text{gc+gR})*[\text{Xp*(gc+gR)+Xx*gI}]$$
$$(\text{gc+gR}) * u_s = (\text{gc+gR})*[\text{Xx*(gc-gR)+Xp*gI}] =$$
$$= \text{Xx} * (\text{gc}^2 - \text{gR}^2) + \text{Xp*gI*(gc+gR)} = \tag{78}$$
$$= \text{Xx*gI*gI+Xp*gI*(gc+gR)} =$$
$$= \text{gI*[Xp*(gc+gR)+Xx*gI]}$$

that is the hard definition apart from a common factor: Xp*(gc+gR)+Xx*gI.